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Positive-Operator-Valued Time Observable in Quantum Mechanics

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We examine the longstanding problem of introducing a time observable in quantum mechanics; using the formalism of positive-operator-valued measures, we show how to define such an observable in a natural way and we discuss some consequences.

1. INTRODUCTION

Since the very beginning of quantum mechanics it has been clear that it is not so easy to define time at a quantum level; in the ordinary theory, in fact, it is not an observable, but an external parameter, in other words, time is *classical*. In considering changing this situation by promoting time to an observable, one has to face a theorem by Pauli (1958) that states, essentially, that such an operator cannot be self-adjoint; since in the usual quantum mechanics, observables are postulated to be self-adjoint operators (see, for example, Von Neumann, 1955, and Prugovečki, 1971), this theorem constitutes a problem.

One of the consequences of this is, for example, that one cannot deduce the Heisenberg uncertainty relation for time and energy from a kinematical point of view because time does not belong to the algebra of observables. In spite of this the relation $\Delta T \cdot \Delta H \geq 1$ is commonly accepted as true and it is derived in some way from dynamical considerations.

The situation is quite unsatisfactory both from a physical point of view and from an epistemological point of view and although it has been investigated in many works (see, for example, Aharonov and Bohm, 1961; Rosen-

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baum, 1969; Olkhovsky *et al.*, 1974; Blanchard and Jadczyk, 1996; Grot *et al.*, 1996), we are unaware of a definitive and satisfactory solution of the problem.

The "problem of time" has some consequences also in the realm of quantum gravity, i.e., in the struggle to give a quantum description of spacetime in order to solve some divergences problems in both general relativity (singularity theorems) and in quantum field theory (renormalization problem). A quantum "spacetime" with zero spatial dimensions and one time dimension (that is, the quantization of time) is the simplest model and we think it is preliminary to any other attempt.

If one adopts the operational point of view (Bridgman, 1927), then defining the concept of time at a quantum level is equivalent to specifying a set of operations useful for the measurement of time; in this context the problem of time is the problem of building "quantum clocks." In this paper we analyze a simple model for such a quantum clock and try to draw some general conclusions on the problem.

2. MATHEMATICAL PRELIMINARIES

Our starting point is a generalized formulation of standard quantum mechanics that extends the usual observable concept. A justification of such a formulation is given by Gleason's theorem (Busch *et al.*, 1991), which guarantees that this structure is the most general one compatible with the probablisitic interpretation of quantum mechanics (Copenhagen interpretation); other justifications come from work by Ludwig (1968) and Giles (1970), but they are beyond the scope of this paper. In this section we summarize, in a very concise and incomplete way, the mathematical tools that we shall use later; for a good review of the subject, along with a very complete bibliography, see Busch *et al.* (1991), Giles (1970), and Davies (1976).

A given quantum system $\mathscr{S}$ is described by a Hilbert space $\mathscr{H}$; we call $\mathscr{L}(\mathscr{H})$ the algebra of bounded operators on $\mathscr{H}$, $\mathscr{L}(\mathscr{H})^+$ the cone of positive ones, and $\mathscr{T}(\mathscr{H})$ the subalgebra of the trace-class operators. The *states* of the system $\mathscr{S}$ are the positive operators with trace one on $\mathscr{H}$ that form a convex set $\mathscr{T}(\mathscr{H})_1^+$ in $\mathscr{T}(\mathscr{H})$.

Given a measurable space $(\Omega, \mathscr{F})$, where Ω is a nonempty set and $\mathscr{F}$ a σ-algebra of subsets of Ω, a *normalized positive-operator-valued measure* (a POV-measure) τ is a map

$$\tau: \quad \mathscr{F} \to \mathscr{L}(\mathscr{H})^+$$

such that:

1. $\tau(X) \geq \tau(\emptyset) = 0, \ \forall X \in \mathscr{F}.$

2. $\tau(\cup X_i) = \Sigma\tau(X_i)$, where $\{X_i\}$ is a countable collection of disjoint elements of $\mathcal{F}$ and the convergence is in the weak topology.
3. $\tau(\Omega) = I$.

If $\tau(X)^2 = \tau(X)$, then τ is a *projection-valued measure* (PV-measure) and it can be demonstrated that this property is equivalent to

$$\tau(X \cap Y) = \tau(X)\tau(Y)$$

If Ω is the real Borel space $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ and τ is a PV-measure, than it is a spectral representation of a unique self-adjoint operator A

$$A = \int_\Omega \lambda\tau(d\lambda) \tag{1}$$

A *generalized observable* is a POV-measure on a particular measurable space, while a PV-measure, via the relation (1), represents an ordinary observable of quantum mechanics. This generalization of the concept of an observable is possible in view of the probabilistic interpretation of quantum mechanics [for more details see Busch *et al.* (1991)]. Given an observable τ and a state ρ, we have a probability measure τ_ρ on $(\Omega, \mathcal{F})$,

$$\tau_\rho: \mathcal{F} \to [0, 1]$$

$$\tau_\rho: X \to \text{Tr}\,[\rho\tau(X)]$$

This can be interpreted as the probability that the measure of the observable τ on the state ρ lies in the set X.

The mean value of the observable τ on the state ρ is then

$$\text{mean}(\tau, \rho) = \int \lambda\tau_\rho(d\lambda)$$

while the variance is given by

$$\text{var}^2(\tau, \rho) = \int\lambda^2\tau_\rho(d\lambda) - (\text{mean}\,(\tau, \rho))^2$$

Let $\mathcal{G}$ be a locally compact group, $(\Omega, \mathcal{F})$ a measurable $\mathcal{G}$-space, and U a unitary representation of $\mathcal{G}$ on a Hilbert space $\mathcal{H}$; if τ is a POV-measure on $(\Omega, \mathcal{F})$ with values in $\mathcal{L}(\mathcal{H})^+$, then we say that τ is *covariant* with respect to U if

$$U_g\tau(X)U_g^* = \tau(X_g)$$

for every $X \in \mathcal{F}$ and every $g \in \mathcal{G}$. The pair (τ, U) is called a *system of covariance* (Davies, 1976); if τ is a PV-measure, then (τ, U) is a *system of imprimitivity* (Mackey, 1963; Varadarajan, 1984).

The condition of covariance means that

$$(U_g\tau(X)U_g^*)_\rho = \tau_{U_g^*\rho U_g}(X)$$

As stated in the introduction, due to an argument by Pauli (1958), it is not possible to have a self-adjoint operator for a time observable in quantum mechanics:

Theorem 1 (Pauli). Given an observable (time) T with the following commutation relation with the Hamiltonian,

$$[H, T] = -i$$

then T cannot be a self-adjoint operator.

In the language of POV-measures, the theorem means that a time observable cannot form a system of imprimitivity with the time translations, but it can still form a system of covariance with them. In fact, Pauli's Theorem is a consequence of the following general proposition:

Proposition 1. If τ is a POV-measure on R and it is covariant with respect to the one-parameter group of translations, then

$$\langle\phi|\tau((a, b])|\phi\rangle > 0, \qquad \forall\phi \in \mathcal{H}$$

for every interval $(a, b]$; this means that τ cannot be a PV-measure.

Proof. For the demonstration of the proposition we can proceed in the following way: suppose that we have a POV-measure τ for the observable time and that it forms a system of covariance with $U = \exp(-i\lambda H)$, where H is the generator of the translations. Suppose that for a given pure state ϕ and a certain interval of the real line $(a, b]$ we have

$$\langle\phi|\tau((a, b])|\phi\rangle = 0$$

Then

$$\langle\phi|\tau((a + \lambda, b - c + \lambda])|\phi\rangle = 0, \qquad \forall\lambda \in [0, c]$$

and, for the covariance property,

$$\langle\phi|e^{-i\lambda H}\tau((a, b - c])e^{i\lambda H}|\phi\rangle = 0$$

and so

$$\langle\phi|e^{-i\lambda H}\sqrt{\tau((a, b - c])}\sqrt{\tau((a, b - c])}e^{i\lambda H}|\phi\rangle = 0$$

for the positivity of τ. Finally we have

$$F(\lambda) \equiv \sqrt{\tau((a, b - c])}e^{i\lambda H}|\phi\rangle = 0, \qquad \forall\lambda \in [0, c]$$

But $F(\lambda)$ is a holomorphic vector-valued function in the upper half of the complex plane that is zero on the interval $[0, c]$; using the *Riemann–Schwarz reflection principle* (Titchmarsh 1939), one can prove that such a function, being zero on an interval, is zero everywhere. This means that $\langle\phi|\tau((a + c$

$- \lambda, b - \lambda])|\phi\rangle$ is zero for all the values of λ, i.e., $\langle\phi|\tau|\phi\rangle$ is zero on all the intervals of R and this is impossible if τ has to be a normalized POV-measure. QED

3. A MODEL FOR A QUANTUM CLOCK

In this section we analyze a particularly simple model for a quantum clock (Rosenbaum, 1969; Toller, 1996) using the mathematical formalism of the preceding section.

Let us consider a one dimensional system represented by the Hilbert space

$$\mathcal{H} = L^2(\mathrm{R})$$

We have, as usual, a coordinate q observable along with its momentum p (in this case, ordinary observables) such that

$$[q, p] = i$$

Moreover, this "clock" has a Hamiltonian equal to

$$H = p^2/2$$

We can interpret q as the time displayed by the clock and p as the rate of the clock itself. In a classical model the real time would be

$$T = q/p$$

but in the quantum case we have to take care of the ordering of the operators. We have to perform an arbitrary choice and we follow Toller (1996), putting

$$T = (2p)^{-1}q + q(2p)^{-1}$$

This operator can be defined on the domain (in the "p-representation") of infinitely differentiable functions over the compact subsets of R $-$ $\{0\}$, which is dense in $\mathcal{H}$ (it is also possible to use as the domain the set of infinitely differentiable functions over the compact subsets of R and then imposing the condition of Hermiticity, which gives $\lim_{p\to 0}[\phi(p)/\sqrt{p}] = 0$, $\forall\phi \in \mathfrak{D}(T)$).

It is easy to see that T is Hermitian and the expected commutation relation

$$[H, T] = -i$$

is satisfied on $\mathfrak{D}(T)$. Now, for the Pauli theorem, we know that T cannot be an ordinary observable, but we can still see if it can be interpreted in the generalized framework of the preceding section. To do so we have to find a POV-measure τ on R such that

$$\langle \phi | T | \phi \rangle = \int_{\mathbf{R}} \lambda \langle \phi | \tau(d\lambda) | \phi \rangle, \qquad \forall | \phi \rangle \in \mathcal{D}(T) \subset \mathcal{H}$$

Moreover, (τ, U) has to be a covariance system with $U = \exp(-i\lambda H)$ a representation of the time-translation group $\mathcal{G}$.

In order to build τ, let us start to search the eigenstates of T; it is convenient to work in the momentum representation instead of the usual coordinate representation (in this way it is simpler to define the operator p^{-1}). In such a representation we have

$$T = i(2p)^{-1} \frac{d}{dp} + i \frac{d}{dp} (2p)^{-1}$$

The eigenvector problem reads

$$T|t\rangle = t|t\rangle$$

and defining the wavefunction $\psi_t(p)$ as

$$\psi_t(p) = \langle p | t \rangle$$

we have

$$T\psi_t(p) = t\psi_t(p)$$

This equation defines a double family of eigenfunctions:

$$\langle p | t, \alpha \rangle = \psi_{t\alpha}(p) = \frac{1}{\sqrt{2\pi}} \theta(\alpha p) \sqrt{|p|} \exp\left(-\frac{itp^2}{2} \right)$$

with $\alpha = \pm 1$. They do not lie in $\mathcal{H}$, and so they have to be regarded as weak eigenfunctions:

$$\langle t, \alpha | (T - t) | \phi \rangle = 0, \qquad \forall \phi \in \mathcal{D}(T)$$

We can also see easily that the eigenvectors of T are not orthogonal

$$\langle t, \alpha | t', \alpha' \rangle = 0 \qquad \text{with} \quad \alpha \neq \alpha'$$

$$\langle t, \alpha | t', \alpha \rangle = \frac{1}{2} \delta(t - t') + \frac{i}{2} P \frac{1}{\pi(t - t')}$$

The following relation still holds (in the weak sense):

$$\sum_{\alpha} \int_{-\infty}^{+\infty} dt \, |t, \alpha\rangle\langle t, \alpha| = \mathbf{1}$$

At this point we can state the following propositions:

Proposition 2. $\tau(dt) = \Sigma_\alpha |t, \alpha\rangle\langle t, \alpha| \, dt$ gives a POV-measure

$$\tau(X) = \int_X \tau(dt) = \sum_\alpha \int_X |t, \alpha\rangle\langle t, \alpha| \, dt$$

with X a Borel set of the real line.

Proposition 3. The system (τ, U), where $U = \exp(-i\lambda H)$ is a representation of the one-parameter group $\mathcal{G}$ of time translations, is a covariance system.

Proof. Let us start from the first one; obviously $\tau(X)$ is a positive operator; moreover, it is bounded,

$$\tau(X) \leq \tau(\mathrm{R}) = \sum_\alpha \int_{-\infty}^{+\infty} dt \, |t, \alpha\rangle\langle t, \alpha| = \mathbf{1}$$

so that $\tau(X) \in \mathcal{L}^+(\mathcal{H})$. The σ-additivity follows from the additivity of integrals and τ is normalized to $\mathbf{1}$.

For the second proposition one can see that

$$e^{i\lambda H}|t, \alpha\rangle = |t - \lambda, \alpha\rangle$$

and so

$$\begin{aligned}
\langle\phi|e^{i\lambda H}\tau(X)e^{-i\lambda H}|\phi\rangle &= \int_X \langle\phi|e^{i\lambda H}\tau(dt)e^{-i\lambda H}|\phi\rangle \\
&= \sum_\alpha \int_X dt \, \langle\phi|e^{i\lambda H}|t, \alpha\rangle\langle t, \alpha|e^{-i\lambda H}|\phi\rangle \\
&= \sum_\alpha \int_X dt \, \langle\phi|t - \lambda, \alpha\rangle\langle t - \lambda, \alpha|\phi\rangle \\
&= \langle\phi|\tau(X - \lambda)|\phi\rangle
\end{aligned}$$

which is the relation of covariance of the POV-measure τ. QED

In conclusion we can say that τ is a generalized observable for the time of our quantum clock; it can be checked that τ is not a PV-measure (essentially this is a consequence of the nonorthogonality of the eigenvectors of T) and so there is no contradiction with the Pauli Theorem.

We have studied a particular POV-measure for a time observable obtained by choosing a very particular time operator; the next step is to study POV-measures for time regardless of operator. The interesting object is the space of POV-measures that form a system of covariance with a representation of time translations; the task is to find in such a space the "best" measures to be used for quantum clocks. This will be the argument of a future paper.

4. UNCERTAINTY RELATIONS

We now can examine the uncertainty relations for time and energy from a kinematical point of view, as stated in the introduction.

If we define, for a Hermitian operator A, the quantity

$$(\sigma_A)^2 = \langle\phi|A^2|\phi\rangle - (\langle\phi|A|\phi\rangle)^2 \qquad \text{with } \langle\phi|\phi\rangle = 1$$

we can prove (von Neumann, 1955) that for the operators T and H of the preceding section (they are Hermitian) the following relation is true on a certain domain of $\mathcal{H}$:

$$\sigma_T \sigma_H \geq 1/2$$

This relation is commonly accepted as the equivalent for time and energy of the famous Heisenberg relation for position and momentum; the fact is that the quantity σ_T is not, in general, the variance of the observable time ΔT, because T is a generalized observable and it is not a self-adjoint operator. But in our simple model the two quantities coincide; in fact we can write

$$T|\phi\rangle = \sum_{\alpha} \int_{-\infty}^{+\infty} dt \, |t, \alpha\rangle\langle t, \alpha|T|\phi\rangle, \qquad \forall|\phi\rangle \in \mathcal{D}(T)$$

for the property of τ exposed in the preceding section; since $|t, \alpha\rangle$ is a weak eigenvector of T, we have

$$T|\phi\rangle = \sum_{\alpha} \int_{-\infty}^{+\infty} dt \, t|t, \alpha\rangle\langle t, \alpha|\phi\rangle$$

From this relation one sees that the mean of T, as defined in Section 2, is the usual one

$$\langle\phi|T|\phi\rangle = \sum_{\alpha} \int_{-\infty}^{+\infty} dt \, t\langle\phi|t, \alpha\rangle\langle t, \alpha|\phi\rangle$$

$$= \int_{-\infty}^{+\infty} t\langle\phi|\tau(dt)|\phi\rangle = \text{mean}(\tau, \phi)$$

Using the relation

$$\langle\phi|T^2|\phi\rangle = \sum_{\alpha} \sum_{\alpha'} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' \, tt'\langle\phi|t', \alpha'\rangle\langle t', \alpha'|t, \alpha\rangle\langle t, \alpha|\phi\rangle$$

we obtain

$$\langle\phi|T^2|\phi\rangle = \frac{1}{2}\int_{-\infty}^{+\infty} t^2\langle\phi|\tau(dt)|\phi\rangle + \Lambda_\phi$$

where

$$\Lambda_\phi = \frac{i}{2\pi}\sum_\alpha \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt'\, \overline{t'\phi(t', \alpha)}t\phi(t, \alpha)P\frac{1}{(t' - t)}$$

with $\phi(t, \alpha) = \langle t, \alpha|\phi\rangle$. One can check that

$$\Lambda_\phi = \frac{1}{2}\int_{-\infty}^{+\infty} t^2\langle\phi|\tau(dt)|\phi\rangle$$

and then

$$\langle\phi|T^2|\phi\rangle = \int_{-\infty}^{+\infty} t^2\langle\phi|\tau(dt)|\phi\rangle$$

In the end we have for the generalized observable τ

$$\sigma_T = \text{var}(\tau, \phi)$$

and so the uncertainty relation for time and energy variances is obtainable in a rigorous way within the POV-measure formalism.

5. CONCLUSIONS

We have shown how it is possible to give a well-defined meaning to the concept of time observable at a quantum level using the POV-measure formalism; in particular we have studied a simple quantum clock model, giving a precise mathematical derivation of the Heisenberg uncertainty relation for time and energy. Since clocks are fundamental in the operational definition of spacetime, we believe this is a preliminary step toward an analysis of spacetime concepts at a quantum level, an analysis that we hope to present in future works.

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